Physics 216

Lecture II - Ideal Fluids

- Equations
- Basic Concepts - especially $\left\{ \begin{array}{l} \text{Kelvin's Thm.} \\ \text{potential Flow} \end{array} \right.$
- Induced Mass, Quasi (Pseudo) Momentum

I.) Euler Equations / Ideal Fluids

"The Flow of Dry Water" - RPF.

$\underline{V}, \sigma, \rho$ Volume V

$\underline{\nabla}\ \underline{S}$ density ρ

argue macroscopically but really derive from Boltzmann Eqn.

- Mass conservation:

$$\frac{dM}{dt} = \frac{\partial}{\partial t}\int d^3x\, \rho(\underline{x},t) = -\int d\underline{S}\,[\underline{V}\rho]$$

$$= -\int d^3x\, \underline{\nabla}\cdot(\rho\underline{V})$$

so

$$\boxed{\partial_t \rho + \underline{\nabla}\cdot(\rho\underline{V}) = 0}$$

continuity

$$\partial_t\rho + \underline{\nabla}\cdot\underline{\Gamma} = 0 \qquad \underline{\Gamma} = \rho\underline{V}$$

mass flux density

- Momentum conservation:

$$\underset{\substack{\downarrow \\ \text{net force density} \\ \text{on element}}}{\underline{f}} = -\underset{\substack{\downarrow \\ \text{pressure} \\ \text{gradient}}}{\underline{\nabla} p} + \underline{F}$$

fluid element

$\underline{F}$ → body force $\left(\frac{\underline{J} \times \underline{B}}{c}\right.$ in MHD$\left.\right)$

so, Sir Isaac:

$$\rho \underset{\substack{\downarrow \\ \text{acceleration}}}{\underline{a}} = -\underline{\nabla} p + \underline{F}$$

$$\underline{a} = \frac{d\underline{v}}{dt}$$ → what does this mean (substantive derivative)

here:

$d\underline{v}$ → increment in $\underline{v}$

$$\underset{\substack{\downarrow \\ \text{velocity} \\ \text{increment}}}{d\underline{v}} = \underset{\substack{\downarrow \\ \text{local} \\ \text{acceleration}}}{\frac{\partial \underline{v}}{\partial t}}\, dt + \underset{\substack{\downarrow \\ \text{particle moves/displaced} \\ \text{in inhomogeneous} \\ \text{velocity field.}}}{\overset{\substack{\text{displ.} \\ \downarrow}}{d\underline{r}} \cdot \underline{\nabla}\, \underline{v}}$$

$$d\underline{v}/dt = \frac{\partial \underline{v}}{\partial t} + \frac{d\underline{r}}{dt} \cdot \underline{\nabla}\, \underline{v}$$

$$= \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}$$

$\Rightarrow$

$$\boxed{\rho \frac{d\underline{v}}{dt} = \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}\right) = -\underline{\nabla} p + \underline{f}}$$

Euler Eqn.

? Momentum Flux ?

$$\partial_t (\rho v_i) = -\frac{\partial}{\partial x_k} \Pi_{ik}$$

i.e.

$$\underbrace{\partial_t (\rho \underline{v})}_{\text{momentum density}} = \underline{v}\, \partial_t \rho + \rho \frac{\partial \underline{v}}{\partial t}$$

$$= -\underline{v}\left(\rho(\underline{\nabla} \cdot \underline{v}) + \underline{v} \cdot \underline{\nabla} \rho\right) + \rho\left(-\underline{v} \cdot \underline{\nabla}\, \underline{v} - \frac{\underline{\nabla} p}{\rho}\right)$$

$$= -\left(\rho\left[\underline{v}(\underline{\nabla} \cdot \underline{v}) + \underline{v} \cdot \underline{\nabla}\, \underline{v}\right] + \underline{v}(\underline{v} \cdot \underline{\nabla} \rho)\right) - \underline{\nabla} p$$

$$= -\underline{\nabla} \cdot \left(\underbrace{\rho \underline{V}\, \underline{V}}_{\substack{\text{Reynolds} \\ \text{stress tensor}}} + \underbrace{\underline{\underline{I}}}_{\text{identity tensor}} P \right)$$

→ analogous to Maxwell stress tensor

so

$$\Pi_{ik} = \rho V_i V_k + \delta_{ik} P$$

and

$$\frac{\partial}{\partial t} \int d^3x\, \rho \underline{V} = \frac{d}{dt} \underline{P} = -\int d\underline{S} \cdot \left(\rho \underline{V}\, \underline{V} + \underline{\underline{I}} P \right).$$

$\Pi_{ik}\, dS_k \equiv$ momentum flux in ith direction.

$$\boxed{\Pi_{ik} = \rho V_i V_k + \delta_{ik} P.}$$

defines momentum flux

- in N.-S. Eqn, viscous stress appears due momentum flux from collisions, interacting with macroscopic flow gradients

→ Mass, Momentum and Energy!

In ideal fluid, no heat exchanged between fluid elements ⇒ motion adiabatic – i.e. entropy conserved along trajectories

$$\frac{ds}{dt} = 0$$

$s \equiv$ entropy per mass.

$$\Rightarrow \boxed{\frac{\partial s}{\partial t} + \underline{v} \cdot \underline{\nabla} s = 0}$$

adiabatic equation for fluid.

For energy flux:

$$\mathcal{E} = \rho \frac{v^2}{2} + \rho \epsilon$$

$\mathcal{E}$ → total energy density of fluid element

$\rho \frac{v^2}{2}$ → kinetic energy density

$\rho\epsilon$ → internal energy density (i.e. thermal).

Now as with momentum, consider $\partial \mathcal{E}/\partial t$,

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial}{\partial t}\left(\underset{①}{\rho \frac{v^2}{2}} + \underset{②}{\rho \epsilon}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\rho \underline{V}^2}{2}\right) = \frac{\underline{V}^2}{2}\frac{\partial \rho}{\partial t} + \rho \underline{V}\cdot\frac{\partial \underline{V}}{\partial t}$$

$$= -\frac{\underline{V}^2}{2}\underbrace{\underline{\nabla}\cdot(\rho \underline{V})}_{\text{cont.}} \underbrace{- \underline{V}\cdot\underline{\nabla} P - \rho \underline{V}\cdot(\underline{V}\cdot\underline{\nabla}\underline{V})}_{\text{momentum balance}}$$

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{V}\times\underline{\omega} + \frac{1}{2}\nabla(V^2)$$

$$\underline{\omega} = \underline{\nabla}\times\underline{V} \rightarrow \text{vorticity}$$

so

$$\rho \underline{V}\cdot(\underline{V}\cdot\underline{\nabla}\underline{V}) = \rho \underline{V}\cdot\left(-\underline{V}\times\underline{\omega} + \underline{\nabla}\left(\frac{V^2}{2}\right)\right)$$

$$= \rho \underline{V}\cdot\underline{\nabla}\frac{V^2}{2}$$

and

$$\underset{\text{enthalpy}}{dW} = dE + d(PV)$$

$$= TdS + VdP$$

$$= TdS + \frac{dP}{\rho}$$

so

$$\underline{\nabla} P = \rho \nabla W - \rho T \underline{\nabla} S.$$

thus, ①

$$\frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right) = -\frac{V^2}{2}\,\underline{\nabla}\cdot(\rho\underline{V}) - \rho\underline{V}\cdot\underline{\nabla}\left(\frac{V^2}{2}+w\right)$$

$$+\rho T\,\underline{V}\cdot\underline{\nabla}s$$

For ②:

$$\frac{\partial}{\partial t}(\rho\epsilon) =$$

useful to transform using thermodynamic identity:

$$d\epsilon = dQ - p\,dV$$
$$= T\,dS - p\,dV$$

but $V = 1/\rho$

$$dV = -d\rho/\rho^2$$

$$d\epsilon = T\,dS + \frac{p}{\rho^2}\,d\rho$$

so $d(\rho\epsilon) = \rho\,d\epsilon + \epsilon\,d\rho$

so

$$d(\rho\epsilon) = \left(\frac{p}{\rho} + \epsilon\right) d\rho + \rho T ds$$

$$\epsilon + \frac{p}{\rho} = \epsilon + pV = w$$

enthalpy

so

$$d(\rho\epsilon) = w\,d\rho + \rho T ds$$

② $$\frac{\partial}{\partial t}(\rho\epsilon) = w\frac{\partial \rho}{\partial t} + \rho T \frac{\partial s}{\partial t}$$

$$= -w\,\underline{\nabla}\cdot(\rho \underline{v}) - \rho T \underline{v}\cdot\underline{\nabla} s$$

So, combining ①, ②:

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \rho\epsilon\right) = -\frac{v^2}{2}\underline{\nabla}\cdot(\rho\underline{v}) - \rho\underline{v}\cdot\underline{\nabla}\left(\frac{v^2}{2} + w\right) - w\,\underline{\nabla}\cdot(\rho\underline{v}) - \rho T\,\underline{v}\cdot\underline{\nabla} s + \rho T\,\underline{v}\cdot\underline{\nabla} s$$

$$= -\left(\frac{v^2}{2} + w\right)\underline{\nabla}\cdot(\rho\underline{v}) - \rho\underline{v}\cdot\underline{\nabla}\left(\frac{v^2}{2} + w\right)$$

$$= -\underline{\nabla}\cdot\left(\rho\underline{v}\left(\frac{v^2}{2} + w\right)\right)$$

Thus have:

$$\frac{\partial}{\partial t}\left(\frac{\rho V^2}{2} + \rho \epsilon\right) + \underline{\nabla}\cdot\left(\rho \underline{V}\left(\frac{V^2}{2} + w\right)\right) = 0$$

so

$$\underbrace{\frac{\partial}{\partial t}\int^{V} d^3x \left(\frac{\rho V^2}{2} + \rho\epsilon\right)}_{\text{change in energy in volume } V} = \underbrace{-\int d\underline{S}\cdot\left[\rho\underline{V}\left(\frac{V^2}{2}+w\right)\right]}_{\text{energy flux density thru surface.}}$$

$$\boxed{\begin{array}{l}\text{energy density flux} \\ \underline{Q} = \rho \underline{V}\left(\frac{V^2}{2} + w\right)\end{array}}$$

accompanies Π_{ik}, $\underline{\Gamma}$.

→ Meaning:

$$w = \epsilon + \frac{P}{\rho}$$

so

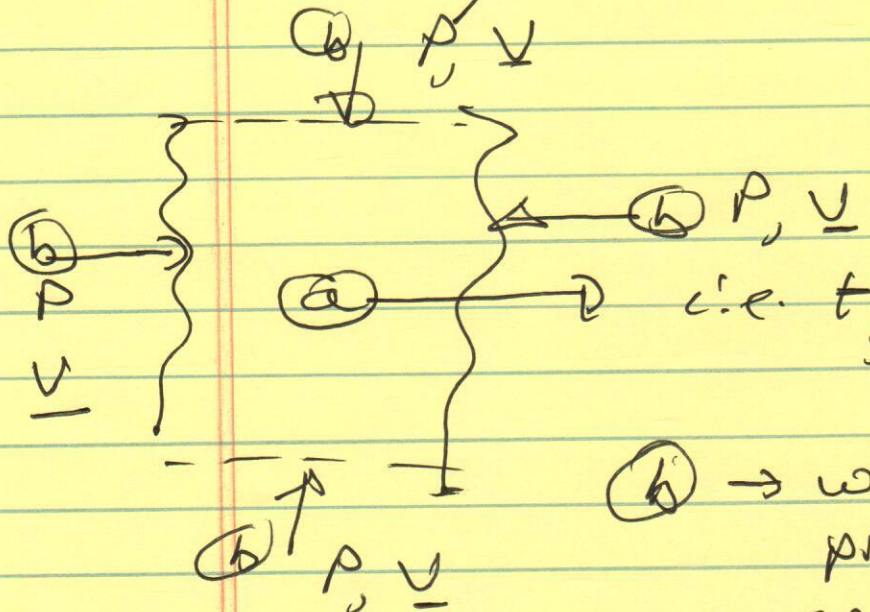
$$\int d\underline{S}\cdot\underline{Q} = \underbrace{\int d\underline{S}\cdot\rho\underline{V}\left(\frac{V^2}{2}+\epsilon\right)}_{\text{(a) flux of KE and internal energy thru surface}} + \underbrace{\int d\underline{S}\,\rho\underline{V}\,\frac{P}{\rho}}_{\text{(b)}}$$

(b) $= \int d\underline{S} \cdot \underline{V}\, P$

→ $P\,d\nabla$ work done by pressure forces on fluid within surface.

$= \int (\underline{V} \cdot d\underline{S})\, P$

$d\nabla / dt$



i.e. transport thru surface

(b) → work done by pressure forces on fluid within S.

II.) Basic Concepts

Now, convenient to note:

$$dE = đQ - p\,dV$$
$$= T\,dS - p\,dV$$

$W = E + pV$ → enthalpy as Legendre transform of entropy

then

$$dW = T\,dS + V\,dp$$
$$= T\,dS + \frac{dp}{\rho}$$

so, for isentropic motions ($dS = 0$),

$$dp/\rho = dW \quad \text{or} \quad \frac{\underline{\nabla} p}{\rho} = \underline{\nabla} W$$

→ has advantage of RHS of Euler Eqn. as perfect derivative,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = \frac{d\underline{v}}{dt} = -\underline{\nabla} W$$

Then, can immediately note:

$$\frac{d}{dt}\oint \underline{v}\cdot d\underline{\ell} = 0$$

→ Circulation conserved for inviscid, isentropic fluid
↓
Kelvin's Thm.

i.e.

$$\frac{d}{dt}\oint \underline{v}\cdot d\underline{\ell} = \oint \frac{d\underline{v}}{dt}\cdot d\underline{\ell} + \oint \underline{v}\cdot \frac{d\underline{\ell}}{dt}$$

$$= \oint \frac{d\underline{v}}{dt}\cdot d\underline{\ell} + \oint \underline{v}\cdot \frac{d}{dt} d\underline{\ell}$$

$$= \oint (-\underline{\nabla} w)\cdot d\underline{\ell} + \oint \underline{v}\cdot d\underline{v}$$

$$= 0 + 0 .$$

i.e.

$$\boxed{\oint_{c} \underline{v}\cdot d\underline{\ell} = \text{const}}$$

for closed contour in ideal, isentropic fluid {note: no use $\underline{\nabla}\cdot\underline{v} = 0$

often:

$$\Gamma = \oint \underline{v}\cdot d\underline{l} \quad \rightarrow \text{ conserved. (absolute)}$$

N.B.: Obvious analogy in mechanics is Poincaré–Cartan invariant.

$$I_{PC} = \oint \underline{p}\cdot d\underline{q} \qquad \frac{d}{dt} I_{PC} = 0$$

for Hamiltonian system.

Now, elementary vector calc. ⇒

normal to enclosed area.

$$\Gamma = \oint_C \underline{v}\cdot d\underline{l} = \int_A \underline{\omega}\cdot d\underline{S} \quad \rightarrow \text{ cons.}$$

$$\underline{\omega} = \nabla\times\underline{v}$$

Vorticity

What is vorticity?

→ describes rotation of fluid element.

→ $\underline{\omega}$ is $2 \times$ effective local angular velocity of fluid

i.e. $\delta \underline{v} = (\underline{\omega} \times \underline{r})/2$.

→ Vorticity is the non-trivial element in fluid dynamics, beyond Bernoulli's Law and potential flow. Vorticity is central to all interesting topics.

Now,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{\nabla} w$$

$$\underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{v} \times (\underline{\nabla} \times \underline{v}) + \underline{\nabla} \frac{\underline{v}^2}{2}$$

$$= -\underline{v} \times \underline{\omega} + \underline{\nabla} \frac{\underline{v}^2}{2}$$

so

$$\frac{\partial \underline{v}}{\partial t} \underbrace{- \underline{v} \times \underline{\omega}}_{\text{Magnus Force}} = -\underline{\nabla}\left(w + \frac{v^2}{2} \right)$$

then $\underline{\nabla} \times$

⇒ Ideal vorticity (Induction) equation:

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{\omega})}$$

$$= -\underline{v}\cdot\underline{\nabla}\,\underline{\omega} + \underline{\omega}\cdot\underline{\nabla}\,\underline{v} - \underline{\omega}\,\underline{\nabla}\cdot\underline{v}$$

$$\boxed{\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{v} - \underline{\omega}\,\underline{\nabla}\cdot\underline{v}}$$

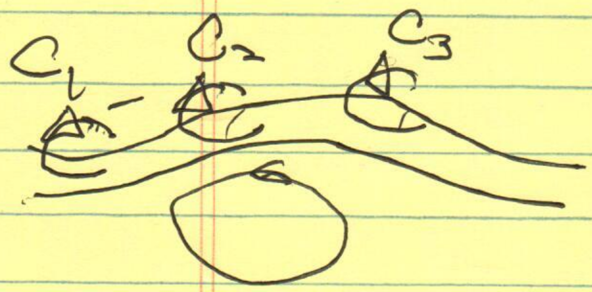
or, with continuity:

$$\boxed{\frac{d}{dt}\,\underline{\omega}/\rho = \underline{\omega}/\rho\cdot\underline{\nabla}\,\underline{v}}$$ → $\underline{\omega}/\rho$ "frozen-in"

→ can derive Kelvin's Thm from induction equation

→ viscosity breaks circulation conservation.

III.) Potential Flow



excludes case of separation!

– Consider fluid streamlines

i.e. streamlines are lines along which fluid flows, i.e.

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z}$$

then if $\underline{\omega} = 0$ at any point on streamline, Kelvin's thm $\Rightarrow$ $\underline{\omega} = 0$ everywhere on line

i.e. Easily seen by considering circulation around infinitesimal loop "pulled" along line. Thus if:

$$\oint_{C_1} \underline{v} \cdot d\underline{l} = \int_{A_1} \underline{\omega} \cdot d\underline{s} = 0 \quad , \text{ then}$$

$$\oint_{C_n} \underline{v} \cdot d\underline{l} = \int_{A_n} \underline{\omega} \cdot d\underline{s} = 0 \quad , \text{ all } C_n$$

- Flow with $\underline{\omega} = 0 = \underline{\nabla} \times \underline{V}$ in all space

  $\Rightarrow$ potential, irrotational flow.

  $\rightarrow$ $\underline{\omega} \neq 0$ $\rightarrow$ vortical rotational